JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 2, 299–332 (1961)

Energy Distribution of Band-Limited Functions Whose Samples on a Half Line Vanish

H. O. POLLAK

Bell Telephone Laboratories, Murray Hill, New Jersey

Submitted by Lotfi Zadeh

Let S be the class of band-limited functions $f(t)$ of bandwidth π such that $\int_{-\infty}^{\infty} f^2(t)\,dt < \infty$ and $f(n) = 0$ for $n = 1, 2, 3, \ldots$. A study is made of the proportion of energy which such a function may have to the right of some given point N. Let

$$\phi(N) = \sup_{f \in S} \frac{\int_N^{\infty} f^2(t)\,dt}{\int_{-\infty}^{\infty} f^2(t)\,dt}.$$

It is shown, among other results, that $\phi(N) = \frac{1}{2}$ if $N \geqslant \frac{3}{4}$, that $\phi(N) > \frac{1}{2}$ if $N \leqslant \frac{1}{4}$, and that $\phi(N) < 1$ for all real N. The first result has the rather surprising consequence that $\phi(N)$ does *not* tend to 0 as $N \to \infty$. It is shown that this conclusion is still valid if we restrict ourselves to finite sampling series; other restrictions on S which do succeed in forcing $\phi(N)$ to 0 for large N are also discussed.

I. INTRODUCTION AND SUMMARY

It is a well-known fact that if $f(t)$ is a band-limited signal which is reasonably well behaved for all time, then $f(t)$ is completely determined by its sample values. Thus, if

$$f(t) = \int_{-\Omega}^{\Omega} F(x)\, e^{ixt}\, dx, \tag{1.1}$$

you can usually expect the relation

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \tag{1.2}$$

to hold. Now the exact conditions under which (1.2) is valid can be somewhat tricky; in our present study, however, we are interested only in the case in which $f(t)$ has finite energy, so that

$$\int_{-\infty}^{\infty} f^2(t)\, dt < \infty$$

In this case, (1.2) is always valid in the mean, and uniformly on compact sets.

Our problem concerns the special subclass of band-limited signals $f(t)$ for which the sample values $f(n\pi/\Omega)$ all vanish for positive n. The question we wish to examine is how the energy of such a signal can be distributed. Since all the nonzero samples are to the left of O, you would also expect much of the energy to be to the left of O. In what sense is this actually true?

Well, there are two elementary observations we can make.

1. Since band-limited functions are necessarily analytic functions of the complex variable t, they cannot vanish in an interval. Consequently it is not possible for *all* the energy to be to the right of O, or of any other point.

2. It is clear that *half* the energy, at any rate, can be to the right of O. The function $(\sin \pi t)/t$ has samples which vanish at the positive sample points, and since it is an even function, it has half its energy to the right of the origin. Are there functions of the required kind with *more* than half their energy to the right of O? Even this is pretty obvious, for consider

$$f(t) = a_0 \frac{\sin \pi t}{t} + a_1 \frac{\sin \pi(t+1)}{t+1}.$$

The functions $(\sin \pi t)/t$ and $[\sin \pi(t+1)]/(t+1)$ are orthogonal over the whole real line $(-\infty, \infty)$ so that the total energy of $f(t)$ is just $\pi^2(a_0{}^2 + a_1{}^2)$. They are *not* orthogonal, however, over just half the real line, $(0, \infty)$, so that $\int_0^\infty f^2(t)\, dt$ involves all three possible terms, and is of the form

$$a_0{}^2 \frac{\pi^2}{2} - 2a_0 a_1 \int_0^{\infty} \frac{\sin^2 \pi t}{t(t+1)}\, dt + a_1{}^2 \int_0^{\infty} \frac{\sin^2 \pi t}{(t+1)^2}\, dt.$$

Since the middle term does not vanish, it is possible, by proper choice of a_0 and a_1, to get *more* than half (by this technique, in fact 54%) of energy to the right of the origin.

This much we can say without doing any work. But now, three questions naturally present themselves:

A. Is there some point N, necessarily to the right of 0, so that we can be sure that *less* than half the energy *must* lie beyond N?

B. If N becomes larger and larger, does the proportion of energy which can be to the right of N become smaller and smaller?

C. Can we be sure that the proportion of energy to the right of the origin, or, for that matter, of any point P, even negative, is *bounded* away from one, or is it possible to find a sequence of functions which come closer and closer to having *all* their energy to the right of P?

In answering these questions, it is convenient to scale the problem so that $\Omega = \pi$, i.e., so that the sample points are at the integers. It is also convenient to have a name for the class of functions under consideration: the set of square-integrable band-limited functions $f(t)$ such that $f(n) = 0$ for $n = 1, 2, 3, \ldots$ will be called S. We can now say the following:

A. The energy to the right of $\frac{3}{4}$ must always be less than half the total energy.

B. No matter how large N is, it is possible to find a band-limited function in S whose energy to the right of N is arbitrarily close to half the total energy. If we use precise mathematical notation, we mean the following: given any $N > 0$, no matter how large, and any $\varepsilon > 0$, no matter how small, there is a function $f(t)$ in S such that

$$\int_N^\infty f^2(t)\,dt > (\tfrac{1}{2} - \varepsilon)\int_{-\infty}^\infty f^2(t)\,dt.$$

C. The proportion of energy to the right of any given point P, even negative, is always bounded away from 1; it is not possible to find a sequence of functions in S whose proportion of energy to the right of P comes arbitrarily close to 1.

Let us discuss these results for a moment. Taken together, (1) and (2) say that, for any $N \geqslant \frac{3}{4}$, the supremum of the proportion of energy to the right of N, when the supremum is taken over all functions in S, is precisely $\frac{1}{2}$. There is, in fact, no function that *attains* $\frac{1}{2}$, but you can come arbitrarily close. On the other hand, we have seen that you can get *more* than half the energy to the right of 0. Where is the break — that is, what is the *largest* N for which it is possible to have as much as half the energy to the right of N? This is still an unsolved problem, it is, however, possible to show that this largest N must be between $\frac{1}{4}$ and $\frac{3}{4}$. In other words, if $N < \frac{1}{4}$, there exist functions in S which have more than half their energy to the right of N.

These results, as results often do, raise several further questions. One is, just what can you say precisely about the energy to the right of 0? In this line, we can only show that the supremum of the possible energies to the right of 0 lies between 0.62 and 0.9 of the total energy. A more interesting question is raised by B. Within the class S, as we have seen, the energy to the right of any point N, no matter how large, does not go to zero, but can stay arbitrarily close to one-half the total energy. This is a surprising result at first sight, and it is natural to ask what more you have to do to *make* the energy to the right of N go to zero as N becomes large. Now the counterexamples which we use to prove B use an infinity of nonzero samples to the left of the origin, and so we wonder what happens if we use only M nonzero samples, at $0, -1, -2, \ldots, -(M-1)$, and require $f(-M)$, $f(-M-1), \ldots$, to vanish along with $f(1), f(2), \ldots$. We can now think, for example, of letting M or N or both become large. What can we say of the energy distribution now?

It turns out that if M and N vary in such a way that the number of samples becomes drastically smaller than the gap — i.e., if $M/N \to 0$ — then the proportion of energy to the right of N does indeed go to zero. But if M/N remains bounded from 0, then the proportion of energy to the right of N need *not* go to zero, and, worst of all, if $M/N \to \infty$, so that you have essentially *more* samples than guard space, then the proportion of energy which it is possible to have to the right of N once again becomes arbitrarily close to $\frac{1}{2}$.

Let us now summarize these results in compact mathematical form. We define $f(t)$ to be in S if

(a) $$f(t) = \int_{-\pi}^{\pi} F(x)\, e^{ixt}\, dx.$$

(b) $$\int_{-\infty}^{\infty} f^2(t)\, dt < \infty.$$

(c) $$f(n) = 0, \qquad n = 1, 2, 3, \ldots .$$

Let

$$\phi(N) = \sup_{f \in S} \frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt}.$$

Then,

THEOREM 1:

$$\phi(N) = \tfrac{1}{2} \qquad \textit{if} \qquad N \geqslant \tfrac{3}{4}.$$

THEOREM 2:

$$\phi(N) < 1 \qquad \textit{for all } N.$$

THEOREM 3:

$$\phi(N) > \tfrac{1}{2} \qquad \textit{if} \qquad N < \tfrac{1}{4}.$$

THEOREM 4:

$$0.62 < \phi(0) < 0.9.$$

If, in addition to the requirements (a), (b), and (c) for membership in S, we demand

(d) $\qquad f(n) = 0, \qquad n = -M, \ -M-1, \ldots,$

then $f(t)$ will be said to be in S_M. Let

$$\phi_M(N) = \sup_{f \in S_M} \frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt}.$$

Then,

THEOREM 5:

$$\left(1 - \frac{1}{N \log \dfrac{M+N}{N}}\right) \frac{1}{2\pi^2} \int_1^{\frac{M}{N+2}+1} \frac{\ln y\,dy}{(y-1)\sqrt{y}} \leqslant \phi_M(N) \leqslant \frac{1}{\pi^2} \int_1^{\sqrt{1+\frac{M+2}{N-2}}} \frac{\ln y\,dy}{(y-1)\sqrt{y}},$$

which is a good estimate if M/N is small. It follows in particular that $\phi_M(N) \to 0$ as $M/N \to 0$, but does not necessarily $\to 0$ if M/N is bounded from 0.

THEOREM 6: *There exists a constant k such that*

$$\frac{1}{2} - \frac{k}{\ln \dfrac{M+N}{N+2}} \leqslant \phi_M(N) \leqslant \frac{1}{2} - \frac{1}{4\pi^2} \sqrt{\frac{N-2}{M+N}},$$

which is a good estimate if M/N is large. It follows in particular that $\phi_M(N) \to \frac{1}{2}$ as $M/N \to \infty$.

A further word of discussion is still warranted. It is an intuitive feeling, Theorems 1 and 6 to the contrary notwithstanding, that if you have a semi-infinite block of zero samples, there should not be much energy far into that block. One way of squirming out of the result of Theorem 1 is squelched by Theorem 6 — keeping the number of nonzero samples finite will not kill the phenomenon if the guard band is to remain bounded. Another possible way of escaping might be to sample a little more frequently than the Nyquist rate — if $f(t)$ has bandwidth Ω, use $f(n\pi/\Omega')$, $\Omega' > \Omega$. If *now* the samples at the positive sample points vanish, can we conclude there is little energy to the right of zero? Yes we can; but unfortunately we have now thrown away too much. There are *no functions* $f(t)$ of bandwidth Ω for which $f(n\pi/\Omega')$, $\Omega' > \Omega$, $n = 1, 2, 3, \ldots$ all vanish, except $f(t)$ identically 0. This is basically Carlson's theorem. It has, in fact, been proved by A. Beurling that the $f(n\pi/\Omega')$ cannot be, in a well-defined sense, too small, even if nonzero.

Yet another attempt at escaping from Theorem 1 can be made by making somewhat more stringent requirements on the integrability of f than just that f be square integrable. Here we have been more successful, and we have a number of results. By no means the sharpest, but perhaps the easiest to interpret is the following:

THEOREM 8: *If $f(t) \in S$, and if in addition the spectrum $F(x)$ of $f(t)$ is of bounded variation, so that $\int_{-\pi}^{\pi} |dF(x)| \leqslant V$, then for $N \geqslant 2$,*

$$\int_N^\infty f^2(t)\,dt < \frac{C \log N}{\sqrt{N}}$$

where C depends only on V and the total energy.

This bound, it appears, does go to zero as N becomes large. This and other theorems with different restrictions on f may be found in the main text.

The author wishes to acknowledge many helpful discussions with his patient colleagues. In particular, the idea for the lower bound of Theorem 4, and for the proof of Theorem 3, was suggested by B. F. Logan; building on Theorem 1, H. J. Landau devised the elegant proof of Theorem 2.

II. MATHEMATICAL PRELIMINARIES

A function $f(t)$ belongs to the class S if

(a) $$f(t) = \int_{-\pi}^{\pi} F(x)\, e^{ixt}\, dx.$$

(b) $$f(t) \in L^2(-\infty, \infty)$$

(c) $$f(n) = 0, \qquad n = 1, 2, 3, \ldots .$$

Some elementary facts about such functions which we shall need are the following:

(d) $f(t)$ may be written in the form

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{\sin \pi t}{t+n}.$$

This is just the sampling theorem with $\pi a_n = (-1)^n f(-n)$.

(e) The total energy in $f(t)$ is $\pi^2 \Sigma_{n=0}^{\infty} a_n^2$. For

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \sum_{n=0}^{\infty} a_n^2 \int_{-\infty}^{\infty} \frac{\sin^2 \pi t}{(t+n)^2}\, dt$$

by the orthogonality of the $(\sin \pi t)/(t+n)$, and the latter integral is independent of n and equals π^2.

(f) The energy in $f(t)$ to the right of N may be written as

$$\begin{aligned}\int_N^{\infty} f^2(t)\, dt &= \int_N^{\infty} \left(\sum_0^{\infty} a_n \frac{\sin \pi t}{t+n}\right)^2 dt \\ &= \int_N^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n a_m \frac{\sin^2 \pi t}{(t+m)(t+n)}\, dt \qquad (2.1) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n a_m \int_N^{\infty} \frac{\sin^2 \pi t}{(t+m)(t+n)}\, dt\end{aligned}$$

by Fubini's theorem, since the series is absolutely convergent. It is clear from this that if $N > 0$, the energy to the right of N is maximized if all the a_n are nonnegative; for if $a_n < 0$, then replacing a_n by $-a_n$ would increase $\int_N^\infty f^2(t)\,dt$ without changing $\int_{-\infty}^\infty f^2(t)\,dt$.

Our problem, then, is to maximize (2.1) under the condition that $\pi^2 \Sigma a_n^2$, the total energy in $f(t)$, be fixed. The mathematical nature of the problem is that of finding or estimating the largest eigenvalue of an infinite matrix (γ_{mn}) where

$$\gamma_{mn} = \int_N^T \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt. \tag{2.2}$$

As motivation for the treatment to come, let us note the following: the expression (2.2) is *almost* equal to

$$h(m, n) = \tfrac{1}{2} \int_N^T \frac{dt}{(t+m)(t+n)},$$

since, on the average, $\sin^2 \pi t = \frac{1}{2}$. If, now, N were equal to 0, then $h(x, y) = \frac{1}{2}\int_0^\infty dt/((t+x)(t+y))$ would satisfy the following conditions:

$$h(x, y) = h(y, x)$$

$$\frac{1}{\sqrt{x}} h(x, 1) \text{ monotone and integrable } (0, \infty)$$

$$\alpha h(\alpha x, \alpha y) = h(x, y).$$

These are precisely the conditions under which the Hilbert Double Series Theorem (see Hardy, Littlewood, and Pólya, ref. [1], Chapter 9) gives the supremum of the eigenvalues of the (h_{mn}) matrix, $m, n \geqslant 1$. Now of course $\sin^2 \pi t$ is *not* $\frac{1}{2}$, and N is *not* 0, and m and n run from 0, not 1, but various forms and proofs of the Hilbert Double Series Theorem have motivated much of our work. In particular, in connection with proving Theorem 6, we shall give a new and sharper form of the Hilbert Double Series Theorem under the stronger assumption that $\sqrt{x}\, h(x, 1)$ is also monotone.

III. Proofs of Theorems 1–4

THEOREM 1: (a) *If* $f(t) \in S$, *then*

$$\int_{3/4}^\infty f^2(t)\,dt \leqslant \tfrac{1}{2} \int_{-\infty}^\infty f^2(t)\,dt.$$

Consequently, the same holds, a fortiori, for $N > \frac{3}{4}$.

(b) *Given any $N > 0$, no matter how large, and any $\delta > 0$, no matter how small, there exists an $f(t) \in S$ such that*

$$\int_N^\infty f^2(t)\,dt > (\tfrac{1}{2} - \delta) \int_{-\infty}^\infty f^2(t)\,dt.$$

These statements together show that, if $N \geqslant \frac{3}{4}$,

$$\sup_{f \in S} \frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} = \tfrac{1}{2}.$$

PROOF: (a) For any integer $B > 0$, let us consider

$$\max \sum_0^B \sum_0^B a_m a_n \int_{3/4}^\infty \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt \tag{3.1}$$

if $\Sigma_0^B a_m{}^2$ is fixed. By the method of Lagrange multipliers, the optimum a_n satisfy

$$\lambda a_n = \sum_{m=0}^B a_m \int_{3/4}^\infty \frac{\sin^2 \pi t\,dt}{(t+m)(t+n)},$$

where λ is the maximum eigenvalue. As we saw before, $a_n \geqslant 0$ for a maximum; in fact $a_n > 0$, for if some $a_n = 0$, then setting $a_n = \delta$ would change $\Sigma a_n{}^2$ by $0(\delta^2)$ while changing (3.1) by $0(\delta)$, which contradicts the assumption of a maximum.

Since $a_n > 0$ for all n, it makes sense to assume that $a_n\sqrt{n + \frac{1}{2}}$ is maximized at $n = \mu$. Then

$$\begin{aligned}
\lambda a_\mu &= \sum_{m=0}^B a_m \int_{3/4}^\infty \frac{\sin^2 \pi t\,dt}{(t+m)(t+\mu)} \\
&\leqslant a_\mu \sqrt{\mu + \tfrac{1}{2}} \sum_{m=0}^B \frac{1}{\sqrt{m + \frac{1}{2}}} \int_{3/4}^\infty \frac{\sin^2 \pi t\,dt}{(t+m)(t+\mu)} \\
&= a_\mu \sqrt{\mu + \tfrac{1}{2}} \int_{3/4}^\infty dt\,\frac{\sin^2 \pi t}{t+\mu} \sum_{m=0}^B \frac{1}{(t+m)\sqrt{m+\frac{1}{2}}}.
\end{aligned}$$

But $1/((t+x)\sqrt{x+\frac{1}{2}})$ is a convex function of x for any $t > 0$, and consequently, for $t > \frac{1}{2}$,

$$\sum_{m=0}^{B} \frac{1}{(t+m)\sqrt{m+\frac{1}{2}}} = \sum_{m=0}^{B} \frac{1}{(t-\frac{1}{2}+m+\frac{1}{2})\sqrt{m+\frac{1}{2}}}$$

$$< \int_0^{B+1} \frac{dx}{(t-\frac{1}{2}+x)\sqrt{x}}$$

$$< \int_0^{\infty} \frac{dx}{(t-\frac{1}{2}+x)\sqrt{x}}$$

$$= \frac{\pi}{\sqrt{t-\frac{1}{2}}}.$$

Therefore,

$$\lambda < \pi\sqrt{\mu+\tfrac{1}{2}} \int_{3/4}^{\infty} \frac{\sin^2 \pi t\, dt}{(t+\mu)\sqrt{t-\frac{1}{2}}}$$

$$= \pi\sqrt{\mu+\tfrac{1}{2}} \int_0^{\infty} \frac{\sin^2 \pi(u+\frac{3}{4})\, du}{(u+\frac{3}{4}+\mu)\sqrt{u+\frac{1}{4}}}$$

$$< \pi\sqrt{\mu+\tfrac{1}{2}} \int_0^{\infty} \frac{\sin^2 \pi(u+\frac{3}{4})\, du}{(u+\mu+\frac{1}{2})\sqrt{u}}$$

But

$$\sin^2 \pi(u+\tfrac{3}{4}) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2\pi u + \tfrac{3}{2}\pi)$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\sin 2\pi u,$$

so that

$$\lambda < \frac{\pi}{2}\sqrt{\mu+\tfrac{1}{2}} \int_0^{\infty} \frac{(1-\sin 2\pi u)\, du}{(u+\mu+\frac{1}{2})\sqrt{u}}.$$

Since $[1/(u+\mu+\frac{1}{2})\sqrt{u}]$ is monotone decreasing,

$$\int_0^\infty \frac{\sin 2\pi u}{(u+\mu+\frac{1}{2})\sqrt{u}}\,du > 0,$$

while

$$\int_0^\infty \frac{du}{(u+\mu+\frac{1}{2})\sqrt{u}} = \frac{\pi}{\sqrt{\mu+\frac{1}{2}}}.$$

Hence

$$\lambda < \frac{\pi^2}{2}.$$

Thus, for any B

$$\sum_{m=0}^{B}\sum_{n=0}^{B} a_m a_n \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt < \frac{\pi^2}{2}\sum a_n^2,$$

so that, letting $B \to \infty$ and remembering $2e$ and $2f$, we obtain

$$\int_{3/4}^{\infty} f^2(t)\,dt \leqslant \tfrac{1}{2}\int_{-\infty}^{\infty} f^2(t)\,dt.$$

(b) We first need to give an estimate for

$$\int_N^\infty \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt$$

from below. We have

$$\begin{aligned}
\int_N^\infty \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt &= \sum_{k=0}^{\infty}\int_{N+k}^{N+k+1} \frac{\sin^2 \pi t\,dt}{(t+m)(t+n)} \\
&= \sum_{k=0}^{\infty}\int_{N}^{N+1} \frac{\sin^2 \pi t\,dt}{(t+k+m)(t+k+n)} \\
&= \sum_{k=0}^{\infty}\left[\int_{N}^{N+1/2} + \int_{N+1/2}^{N+1}\right] \frac{\sin^2 \pi t\,dt}{(t+k+m)(t+k+n)} \\
&> \sum_{k=0}^{\infty}\int_{N+1/2}^{N+1} \frac{dt}{(t+k+m)(t+k+n)},
\end{aligned}$$

since $\sin^2 \pi t + \sin^2 \pi(t + \frac{1}{2}) = 1$. Thus

$$\int_N^\infty \frac{\sin^2 \pi t}{(t+m)(t+n)}\, dt > \frac{1}{2} \sum_{k=0}^{\infty} 2 \int_{N+1/2}^{N+1} \frac{dt}{(t+k+m)(t+k+n)}$$

$$\geqslant \frac{1}{2} \sum_{k=0}^{\infty} \int_{N+1/2}^{N+3/2} \frac{dt}{(t+k+m)(t+k+n)}$$

$$= \frac{1}{2} \int_{N+1/2}^{\infty} \frac{dt}{(t+m)(t+n)}.$$

Let M be the smallest integer $\geqslant N + \frac{1}{2}$. Then

$$\int_N^\infty f^2(t)\, dt > \frac{1}{2} \sum_0^\infty \sum_0^\infty a_n a_n \int_{N+1/2}^{\infty} \frac{dt}{(t+m)(t+n)}$$

$$\geqslant \frac{1}{2} \sum_0^\infty \sum_0^\infty a_m a_n \int_M^\infty \frac{dt}{(t+m)(t+n)}$$

$$= \frac{1}{2} \sum_M^\infty \sum_M^\infty a_{m-M}\, a_{n-M} \int_0^\infty \frac{dt}{(t+m)(t+n)}.$$

Now let $a_{m-M} = m^{-(1+\varepsilon)/2}$. Then

$$\frac{1}{\varepsilon M^\varepsilon} < \int_M^\infty x^{-1-\varepsilon}\, dx < \sum_M^\infty a_{m-M}^2 < \frac{1}{\varepsilon M^\varepsilon} + \frac{1}{M^{1-\varepsilon}} = \frac{1}{\varepsilon M^\varepsilon}\bigl(1 + \varepsilon O(1)\bigr).$$

$$\therefore \sum_M^\infty a_{m-M}^2 = \frac{1}{\varepsilon M^\varepsilon}\bigl(1 + \varepsilon O(1)\bigr),$$

and

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \frac{\pi^2}{\varepsilon M^\varepsilon}\bigl(1 + \varepsilon O(1)\bigr).$$

On the other hand,

$$\int_N^\infty f^2(t)\,dt > \tfrac{1}{2}\sum_M^\infty \sum_M^\infty a_{m-M}\,a_{n-M}\int_0^\infty \frac{dt}{(t+m)(t+n)}$$

$$= \tfrac{1}{2}\sum_M^\infty \sum_M^\infty m^{-\frac{1+\varepsilon}{2}}\,n^{-\frac{1+\varepsilon}{2}}\int_0^\infty \frac{dt}{(t+m)(t+n)}$$

$$> \tfrac{1}{2}\int_M^\infty \int_M^\infty x^{-\frac{1+\varepsilon}{2}}\,y^{-\frac{1+\varepsilon}{2}}\int_0^\infty \frac{dt}{(t+x)(t+y)}\,dy\,dx$$

by monotonicity in x of $x^{-(1+\varepsilon)/2}(t+x)$ for every t. Thus

$$\int_N^\infty f^2(t)\,dt > \tfrac{1}{2}\int_M^\infty \int_M^\infty x^{-(1+\varepsilon)/2}\,y^{-(1+\varepsilon)/2}\,\frac{\ln y/x}{y-x}\,dy\,dx.$$

If $y = ux$, we have

$$\int_N^\infty f^2(t)\,dt > \tfrac{1}{2}\int_M^\infty x^{-(1+\varepsilon)}\int_{M/x}^\infty u^{-(1+\varepsilon)/2}\,\frac{\ln u}{u-1}\,du\,dx$$

$$= \tfrac{1}{2}\int_M^\infty x^{-(1+\varepsilon)}\left(\int_0^\infty - \int_0^{M/x}\right) u^{-(1+\varepsilon)/2}\,\frac{\ln u}{u-1}\,du\,dx.$$

Now

$$\int_0^\infty u^{-(1+\varepsilon)/2}\,\frac{\ln u}{u-1}\,du = \frac{\pi^2}{\sin^2((1-\varepsilon)/2)\pi} = \pi^2 + o(1).$$

On the other hand, for any $\alpha > 0$,

$$\frac{u^\alpha \ln u}{u-1}$$

is *bounded* for all u. If $\alpha = \frac{1}{4} - \varepsilon$, then

$$\int_0^{N/x} u^{-(1+\varepsilon)/2}\,\frac{\ln u}{u-1}\,du < M_0\int_0^{N/x} u^{-3/4}\,du = M_1\,x^{-1/4}.$$

Then

$$\int_N^\infty f^2(t)\,dt > \tfrac{1}{2}\int_M^\infty x^{-(1+\varepsilon)}[\pi^2 + o(1) - M_1 x^{-1/4}]\,dx$$

$$= \tfrac{1}{2}\left[(\pi^2 + o(1))\frac{1}{\varepsilon M^\varepsilon} - M_1\int_M^\infty x^{-(1+\varepsilon)}\,x^{-1/4}\,dx\right]$$

$$= \tfrac{1}{2}\left[(\pi^2 + o(1))\frac{1}{\varepsilon M^\varepsilon} - M_2\right]$$

Thus, for this choice of the a_n,

$$\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} > \tfrac{1}{2}\,\frac{(\pi^2 + o(1))\,(1/\varepsilon M^\varepsilon) - M_2}{(\pi^2/\varepsilon M^\varepsilon)\,1 + \varepsilon O(1)}$$

$$= \tfrac{1}{2} - o(1).$$

If we pick ε sufficiently small, this can be made larger than $\frac{1}{2} - \delta$ for any given δ.

We now know that, if $N \geqslant \frac{3}{4}$,

$$\sup_{f\in S}\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} = \tfrac{1}{2}.$$

In pursuing information for other values of N, we should first like to know whether for N sufficiently negative, this supremum can ever be 1, or whether it is bounded away from 1 for any finite N. The answer is contained in:

THEOREM 2: *For any N whatever,*

$$\sup_{f\in S}\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} = \lambda(N) < 1.$$

Furthermore, if $\lambda(N) > \frac{1}{2}$, it can be attained by a function in S.

PROOF: The proof is accomplished through the theory of linear transformations in Hilbert Space. We summarize the relevant facts (the page references are to Riesz-Nagy [2]).

Our Hilbert Space is the space of square-integrable functions on $(-\infty, \infty)$. The inner product is

$$(f, g) = \int_{-\infty}^{\infty} f\bar{g}\, dt,$$

so that

$$||f||^2 = \int_{-\infty}^{\infty} |f|^2\, dt$$

is just the total energy. Let A be a linear transformation of $L^2(-\infty, \infty)$ into itself. We make the following definitions:

A is *symmetric* if $(Af, g) = (f, Ag)$ for any $f\, g \in L^2(-\infty, \infty)$, (p. 228).

A is *completely continuous* if for every $\{f_n\}$ such that $||f_n|| \leqslant C$, $\{Af_n\}$ contains a subsequence converging in norm to an element of L^2, (p. 178).

A is *bounded* if $||Ah|| \leqslant M||h||$, with M independent of h, (p. 149).

To every symmetric linear transformation we can associate a real pointset called its *spectrum*. We need not go into the general definition; suffice it to say that the spectrum contains all eigenvalues of the transformation (these are often called the *point spectrum*), as well as limit points of the point spectrum, and another portion called the continuous spectrum (together called the *limit points of the spectrum*). If

$$m||f||^2 \leqslant (Af, f) \leqslant M||f||^2,$$

then the spectrum of A lies between m and M. And if A is completely continuous and symmetric, then its spectrum is a pure point spectrum, consisting of the eigenvalues of A, plus possibly an accumulation point there of at 0.

We will use, in addition to these elementary remarks about spectra, the following Theorem: (p. 367). *If a completely continuous symmetric transformation B is added to a bounded symmetric transformation A, the set of limit points of the spectrum remains invariant.*

These concepts apply to our problem as follows: let D be *time-limiting* of a function $f(t)$ to (k, L):

$$Df(t) = \begin{cases} f(t) & \text{if} \quad k < t < L. \\ 0 & \text{otherwise} \end{cases}$$

Let G be *band-limiting* of a function $f(t)$ to $(-\pi, \pi)$: if

$$f(t) = \int_{-\infty}^{\infty} F(x)\, e^{ixt}\, dx,$$

then

$$Gf = \int_{-\pi}^{\pi} F(x)\, e^{ixt}\, dx.$$

Let σ be the operation of first band-limiting, (i.e., applying G) and then discarding the positive samples. If, in particular, f is already band-limited, so that

$$f(t) = \sum_{-\infty}^{\infty} a_n \frac{\sin \pi t}{t + n}.$$

then

$$\sigma f = \sum_{0}^{\infty} a_n \frac{\sin \pi t}{t + n}.$$

Each of the above transformations has the property that it is *bounded* by 1, for each may discard part of the energy of a function, but none can *gain* any energy. For any $f \in L^2$,

$$||Df|| \leqslant ||f||,$$
$$||Gf|| \leqslant ||f||,$$
$$||\sigma f|| \leqslant ||f||.$$

Furthermore, D, G, and σ are each *symmetric*, since they are *projection* operators.

$$(Df, h) = \int_{-\infty}^{\infty} Df \cdot h\, dt = \int_{k}^{L} fh\, dt = \int_{-\infty}^{\infty} f \cdot Dh\, dt = (f, Dh),$$

and the same argument in the Transform plane, together with the Parseval theorem, gives that

$$(Gf, h) = (f, Gh).$$

Now

$$\begin{aligned}(\sigma f, h) &= (G\sigma Gf, h) \text{ since } \sigma G \text{ is already band-limited}\\ &= (\sigma Gf, Gh) \text{ by the symmetry of } G\\ &= (\sigma Gf, \sigma Gh) \text{ by the orthogonality of } \frac{\sin \pi t}{t+n}\\ &= (f, \sigma h) \text{ by retracing the previous steps.}\end{aligned}$$

Theorem 1, stated in this new language, says that if $k \geqslant \frac{3}{4}$ and $L = \infty$,

$$||D\sigma f||^2 \leqslant \tfrac{1}{2}||\sigma f||^2.$$

The projection σ is, in turn, bounded by 1, and we have that $\sigma D\sigma$ is a bounded *symmetric* transformation for which

$$||\sigma D\sigma f|| \leqslant \frac{1}{\sqrt{2}}||f||,$$

and $1/\sqrt{2}$ is the best possible constant.

If, on the other hand, k and L are *both finite*, then $D\sigma$ is a completely continuous operator. For let $\{f_n\}$ be a sequence of functions in L^2 such that $||f_n|| \leqslant C$, and let $h_n = \sigma f_n$. Then $||h_n|| \leqslant C$. But h_n are entire functions of exponential type, and are given in terms of their Fourier transforms, by

$$h_n = \int_{-\pi}^{\pi} H_n(x)\, e^{ixt}\, dx.$$

Then

$$\begin{aligned}|h_n|^2 &< \int_{-\pi}^{\pi} |H_n(x)|^2\, dx \int_{-\pi}^{\pi} |e^{ixt}|^2\, dx\\ &= 2\pi||h_n||^2 \int_{-\pi}^{\pi} |e^{ix \operatorname{Im} t}|^2\, dx\\ &< 4\pi^2 C\, e^{2\pi|\operatorname{Im} t|}.\end{aligned}$$

In any compact set in the t-plane, therefore, the h_n are a family of uniformly bounded analytic functions, and thus form a *normal* family.

In particular, this is true on the compact set (k, L). Thus there exists a subsequence n_j such that the

$$h_{n_j} = \sigma f_{n_j}$$

are uniformly convergent in (k, L). They therefore also converge in L^2-norm on (k, L), and thus there exists a function ϕ in L^2 such that

$$\int_k^L |\phi - h_{n_k}|^2 \, dt \to 0.$$

But this is the same thing as saying that

$$\|D\phi - D\sigma f_{n_k}\| \to 0,$$

so that $D\sigma$ is completely continuous if (k, L) is finite. We require a completely continuous *symmetric* operator, and so we pick $\sigma D\sigma$.

We now apply the previously quoted theorem on the sum of two operators. We have proved that, for any finite k, L, $\sigma D(k, L)\sigma$ is completely continuous, while $\sigma D(L, \infty)\sigma$ is bounded by $1/\sqrt{2}$ if $L \geqslant \frac{3}{4}$. The sum of these two operators is $\sigma D(k, \infty)\sigma$; the theorem says that the only limit points of the spectrum of $\sigma D(k, \infty)\sigma$ are limit points of either $\sigma D(k, L)\sigma$ or $D(L, \infty)\sigma$. But $\sigma D(k, L)\sigma$ is completely continuous, and the only limit point of its spectrum is 0. $\sigma D(L, \infty)\sigma$ is bounded between 0 and $1/\sqrt{2}$, and so any limit point of its spectrum is also between 0 and $1/\sqrt{2}$. But this means that any points in the spectrum of $\sigma D(k, \infty)\sigma$ which are larger than $1/\sqrt{2}$ are isolated, i.e., are in the *point* spectrum, and there can be no accumulation point of such isolated points beyond $1/\sqrt{2}$. But they must all be < 1, since $\sigma D(k, \infty)\sigma$ must lose energy. Hence there can be only a finite number of eigenvalues (that is what the points in the *point* spectrum are) beyond $1/\sqrt{2}$, and hence the largest is bounded away from 1.

But now suppose that

$$\frac{(D\sigma f, D\sigma f)}{(f, f)} = \mu^2,$$

for some f such that $f = \sigma f$. Then, since D is a projection,

$$\frac{(D\sigma f, \sigma f)}{(f, f)} = \mu^2$$

as well, and since σ is a projection,

$$\frac{(\sigma D\sigma f, f)}{(f, f)} = \mu^2.$$

Therefore μ cannot exceed the largest eigenvalue of $\sigma D\sigma$, which we have just proved to exist and be bounded away from 1. Therefore μ itself is bounded away from 1 by $\sqrt{\lambda(k)}$, or

$$\int_k^\infty f^2(t)\,dt \leqslant \lambda(k)\int_{-\infty}^\infty f^2(t)\,dt,$$

which is what we wanted to prove.

THEOREM 3: *If $N < \frac{1}{4}$, there exist $f \in S$ such that*

$$\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} > \tfrac{1}{2}.$$

PROOF: If $N < 0$, $(\sin \pi t)/t$ itself furnishes an example, so that we need to examine only $0 \leqslant N < \frac{1}{4}$. Consider the function

$$f(u) = \frac{1}{\Gamma(1-2M+u)\Gamma(1-u)}$$
$$= \frac{1}{\Gamma(1-M+(u-M))\Gamma(1-M-(u-M))},$$

so that $f(M+t) = f(M-t)$.

The Fourier transform of

$$g(t) = \frac{1}{\Gamma(\mu+t)\Gamma(\mu-t)}$$

is

$$G(x) = \begin{cases} \dfrac{1}{\Gamma(2\mu-1)}\left[2\cos\dfrac{x}{2}\right]^{2\mu-2} & \text{if} \quad |x| < \pi \\ 0 & \text{if} \quad |x| > \pi, \end{cases}$$

and if $\mu = 1 - M$,

$$f(t) = g(t-M).$$

Thus the Fourier transform $F(x)$ of $f(t)$ is just a phase factor e^{iMx} times $G(x)$.

Now $f(t) \in L^2$ if and only if $F(x) \in L^2$ if and only if $G(x) \in L^2$. But $\cos x/2$ vanishes only at $\pm\pi$, and is positive for $|x| < \pi$. Thus $G(x) \in L^2$

if $-2(2\mu - 2) < 1$, i.e., if $\mu > \frac{3}{4}$. This means that $f(t) \in L^2$ if $M < \frac{1}{4}$. From the definition of $G(x)$ it is clear that $f(t)$ is band-limited with bandwidth π, and the factor $1/\Gamma(1-\mu)$ ensures that $f(n) = 0$, $n = 1, 2, 3, \ldots$. Hence $f \in S$. Since f is symmetric around M, however, it has half its energy to the right of M; if we pick $N < M < \frac{1}{4}$, we will have a function with more than half its energy to the right of N. This proves Theorem 3.

For the particular case $N = 0$, we can use any M, $0 < M < \frac{1}{4}$. Then, if $\mu = 1 - M$,

$$\int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{\pi}\frac{1}{\Gamma^2(2\mu-1)}\int_{-\pi}^{\pi}\left(2\cos\frac{y}{2}\right)^{4\mu-4} dy$$
$$= \frac{\Gamma(4\mu-3)}{\Gamma^4(2\mu-1)}.$$

We now wish to estimate

$$\int_0^{\infty} f^2(t)\,dt = \int_0^{1-\mu} f^2(t)\,dt + \int_{1-\mu}^{\infty} f^2(t)\,dt$$
$$= \int_0^{1-\mu} f^2(t)\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} f^2(t)\,dt$$
$$= \int_0^{1-\mu} g^2(t)\,dt + \tfrac{1}{2}\int_{-\infty}^{\infty} f^2(t)\,dt.$$

Now

$$g(0) = \frac{1}{\Gamma^2(\mu)}$$
$$g(1-\mu) = \frac{1}{\Gamma(2\mu-1)},$$

and $g(t)$ is concave in $(0, 1-\mu)$. The latter follows from the Fourier transform representation

$$g(t) = \frac{1}{2\pi}\frac{1}{\Gamma(2\mu-1)}\int_{-\pi}^{\pi}\left[2\cos\frac{x}{2}\right]^{2\mu-2} e^{ixt}\,dx$$
$$= \frac{1}{\pi}\frac{1}{\Gamma(2\mu-1)}\int_0^{\pi}\left[2\cos\frac{x}{2}\right]^{2\mu-2} \cos xt\,dx$$

so that

$$g''(t) = -\frac{1}{\pi}\frac{1}{\Gamma(2\mu-1)}\int_0^\pi \left[2\cos\frac{x}{2}\right]^{2\mu-2} x^2\cos xt\,dx.$$

If $t < \frac{1}{2}$, $\cos xt > 0$ in $(0, \pi)$, so that $g''(t) < 0$. Since $1 - \mu < \frac{1}{4}$, $g(t)$ is concave in $(0, 1-\mu)$.

Hence $g(t) \geqslant g(0) + t/(1-\mu)\,[g(1-\mu) - g(0)]$, so that

$$\int_0^{1-\mu} g^2(t)\,dt > \tfrac{1}{3}(1-\mu)\frac{\Gamma^{-6}(\mu) - \Gamma^{-3}(2\mu-1)}{\Gamma^{-2}(\mu) - \Gamma^{-1}(2\mu-1)}.$$

The maximum value of this, at $\mu = 0.83$, is 0.620, so that we have proved the first part Theorem 4:

$$0.62 \leqslant \sup_{f\in S}\frac{\int_0^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} < 0.9.$$

To prove the second part of Theorem 4, we begin, as in Theorem 1, with Lagrange multipliers, and we find that the eigenvalues λ satisfy

$$\lambda a_m = \sum_{n=0}^\infty a_n \int_0^\infty \frac{\sin^2\pi t\,dt}{(t+m)\,(t+n)}, \qquad m = 0, 1, 2, \ldots .$$

Because there is no gap between the sample points and the lower limit of the integration, the previous method does not work immediately. Instead, we solve for a_0:

$$a_0 = \frac{1}{\lambda - \frac{1}{2}\pi^2}\sum_{n=1}^\infty a_n \int_0^\infty \frac{\sin^2\pi t\,dt}{t(t+n)},$$

and if $a_n\sqrt{n}$, $n \geqslant 1$, is maximized at μ, we have as before,

$$a_0 \leqslant \frac{a_\mu\sqrt{\mu}}{\lambda - \frac{1}{2}\pi^2}\pi^2.$$

Then

$$\lambda a_\mu = a_0 \int_0^\infty \frac{\sin^2 \pi t \, dt}{t(t+\mu)} + \sum_{n=1}^{\infty} a_n \int_0^\infty \frac{\sin^2 \pi t \, dt}{(t+n)(t+\mu)} .$$

Our previous method works on the second term of the above, (because there is now a gap of 1 between samples and integration) and we obtain that the second term is bounded by $(\pi^2/2)a_\mu$. Thus

$$\lambda \leqslant \frac{\pi^2}{\lambda - \frac{1}{2}\pi^2} \sqrt{\mu} \int_0^\infty \frac{\sin^2 \pi t \, dt}{t(t+\mu)} + \frac{\pi^2}{2} .$$

But, for all μ,

$$\frac{\sqrt{\mu}}{t+\mu} \leqslant \frac{1}{2\sqrt{t}},$$

therefore,

$$\lambda \leqslant \frac{\pi^2}{\lambda - \frac{1}{2}\pi^2} \int_0^\infty \frac{\sin^2 \pi t \, dt}{t^{3/2}} + \frac{\pi^2}{2}$$

$$= \frac{\pi^2}{\lambda - \frac{1}{2}\pi^2} \cdot \frac{\pi}{2} + \frac{\pi^2}{2} .$$

We are only interested in $\lambda > \pi^2/2$: then

$$\left(\lambda - \frac{\pi^2}{2}\right)^2 \leqslant \frac{\pi^3}{2}$$

$$\lambda \leqslant \frac{\pi^2}{2}\left(1 + \sqrt{\frac{2}{\pi}}\right) < 0.9\,\pi^2.$$

Since $\int_{-\infty}^{\infty} f^2 \, dt = \Sigma a_n{}^2 \pi^2$, this proves the upper bound.

IV. Functions under Further Restrictions

The remaining effort in this paper concerns attempts to get around Theorem 1. That the percentage of energy far away from any nonzero sample values should not go to zero defies the intuition, and there are various attempts one can make to restrict S further. We shall discuss two classes of such restrictions: Use only finite instead of infinite sampling series, and place stronger integrability conditions than L^2 on either the function or its spectrum.

Let S_M be the set of functions f which are members of S and for which, in addition, $f(n) = 0$ if $n = -M-1, -M-2, \ldots$; i.e., finite sampling series of length $M+1$, with nonzero samples between 0 and $-M$. We wish to study

$$\sup_{f \in S_M} \frac{\int_N^\infty f^2(t)\, dt}{\int_{-\infty}^\infty f^2(t)\, dt},$$

particularly as N gets large, and see how this depends on the limiting value of M/N. Our chief tool, a new form of the Hilbert Double Series Theorem, is no harder to prove in general than for our particular kernel, and we do so.

THEOREM: *Let $k(x, y)$ be a function defined for $x > 0$, $y > 0$, and having the following properties*:

(1) $k(x, y) > 0$.

(2) $k(x, y) = k(y, x)$.

(3) k is homogeneous of degree -1, i.e., $k(x, y) = \alpha k(\alpha x, \alpha y)$ for any $\alpha > 0$.

(4) $k(x, 1)/\sqrt{x}$ is convex, and integrable on $(0, \infty)$.

(5) $\sqrt{x}\, k(x, 1)$ is nonincreasing for $x > 1$.

Then, if $A \geqslant 1$,

$$\sum_{n=A}^{B} \sum_{m=A}^{B} a_m a_n k(m, n) < 2 \int_1^{\sqrt{(B+1/2)/(A-1/2)}} k(y, 1) \frac{dy}{\sqrt{y}} \cdot \sum_A^B a_n^2.$$

PROOF: As in the proof of Theorem 1(a), we use Lagrange multipliers, and we find that

$$\lambda a_n = \sum_{m=A}^{B} a_m k(m, n), \qquad n = A, A+1, \ldots, B.$$

The optimum a_n are all > 0. Let μ be a value of n which maximizes $a_n\sqrt{n}$. Then

$$\lambda a_\mu \leqslant \sum_{m=A}^{B} a_\mu \sqrt{\mu}\, \frac{k(m, \mu)}{\sqrt{m}},$$

so that

$$\lambda \leqslant \sqrt{\mu} \int_{A-1/2}^{B+1/2} k(x, \mu)\, \frac{dx}{\sqrt{x}}$$

by the convexity assumption (4). This, in turn, implies

$$\lambda \leqslant \int_{(A-1/2)/\mu}^{(B+1/2)/\mu} k(x, 1)\, \frac{dx}{\sqrt{x}}$$

by the homogeneity assumption (3). We shall see in a moment that the right-hand side of (4.1) is maximized if μ is the geometric mean of $A - \frac{1}{2}$ and $B + \frac{1}{2}$. If we temporarily assume this to have been already proved, then

$$\lambda \leqslant \int_{\sqrt{(A-1/2)/(B+1/2)}}^{\sqrt{(B+1/2)/(A-1/2)}} k(x, 1)\, \frac{dx}{\sqrt{x}}$$

$$= 2 \int_{1}^{\sqrt{(B+1/2)/(A-1/2)}} k(x, 1)\, \frac{dx}{\sqrt{x}},$$

since

$$k(x, 1)\, \frac{dx}{\sqrt{x}} = k(x, 1)\sqrt{x}\, \frac{dx}{x} = k\left(x, x \cdot \frac{1}{x}\right) \frac{x}{\sqrt{x}} \frac{dx}{x}$$

$$= k\left(1, \frac{1}{x}\right) \frac{1}{\sqrt{x}} \frac{dx}{x} \text{ by homogeneity}$$

$$= -k(y, 1)\sqrt{y}\, \frac{dy}{y} \qquad \text{if } y = \frac{1}{x}, \text{ by symmetry.}$$

It remains to prove that if $A \leqslant \mu \leqslant B$, the maximum of

$$H(\mu) = \int_{(A-1/2)/\mu}^{(B+1/2)/\mu} k(x, 1)\, \frac{dx}{\sqrt{x}}$$

occurs at $\mu = \sqrt{(A - \frac{1}{2})(B + \frac{1}{2})}$. But

$$H'(\mu) = \frac{1}{\mu}\left[k\left(\frac{A - \frac{1}{2}}{\mu}, 1\right)\sqrt{\frac{A - \frac{1}{2}}{\mu}} - k\left(\frac{B + \frac{1}{2}}{\mu}, 1\right)\sqrt{\frac{B + \frac{1}{2}}{\mu}}\right].$$

Thus $H'(\mu) = 0$ if $\mu = \sqrt{(A - \frac{1}{2})(B + \frac{1}{2})}$; assumption (5), that $\sqrt{x}\, k(x, 1)$ is nonincreasing for $x > 1$, together with $\sqrt{(1/x)}k(1/x, 1) = \sqrt{x}\, k(x, 1)$, gives that $H'(\mu) < 0$ if $\mu > \sqrt{(A - \frac{1}{2})(B + \frac{1}{2})}$ and $H'(\mu) > 0$ if $\mu < \sqrt{(A - \frac{1}{2})(B + \frac{1}{2})}$. Thus $\mu = \sqrt{(A - \frac{1}{2})(B + \frac{1}{2})}$ gives a true maximum. [Note: (5) does not follow from (1)–(4), so that it does require a separate assumption.

$$k(x, y) = \begin{matrix} \dfrac{1}{x(1-\varepsilon) + \varepsilon y} & \text{if} & x < y \\ \dfrac{1}{y(1-\varepsilon) + \varepsilon x} & \text{if} & x > y \end{matrix}$$

provides a counterexample if $0 < \varepsilon < \frac{1}{2}$.]

In order to apply this result to our case, we first observe that

$$\int_N^\infty \left(\sum_{n=0}^M a_n \frac{\sin \pi t}{t + n}\right)^2 dt < \tfrac{1}{2} \int_{N-1/2}^\infty \sum_{n=0}^M \sum_{n=0}^M a_n a_n \frac{dt}{(t + m)(t + n)}.$$

This is true since

$$\begin{aligned}
\int_N^\infty \frac{\sin^2 \pi t}{(t + m)(t + n)}\, dt &= \sum_{k=0}^\infty \int_{N+k}^{N+k+1} \frac{\sin^2 \pi t\, dt}{(t + m)(t + n)} \\
&= \sum_{k=0}^\infty \int_N^{N+1} \frac{\sin^2 \pi t\, dt}{(t + k + m)(t + k + n)} \\
&= \sum_{k=0}^\infty \left(\int_N^{N+1/2} + \int_{N+1/2}^{N}\right) \frac{\sin^2 \pi t\, dt}{(t + k + m)(t + k + n)} \\
&= \sum_{k=0}^\infty \int_N^{N+1/2} \left\{\frac{\sin^2 \pi t}{(t + k + m)(t + k + n)}\right. \\
&\qquad \left. + \frac{\sin^2 \pi(t + \frac{1}{2})}{(t + k + m + \frac{1}{2})(t + k + n + \frac{1}{2})}\right\} dt \\
&< \sum_{k=0}^\infty \int_N^{N+1/2} \frac{dt}{(t + k + m)(t + k + n)},
\end{aligned}$$

since $\sin^2 \pi t + \sin^2 \pi(t + \frac{1}{2}) = 1$

$$= \tfrac{1}{2} \sum_{k=0}^{\infty} 2 \int_{N}^{N+1/2} \leqslant \tfrac{1}{2} \sum_{k=0}^{\infty} \int_{N-1/2}^{N+1/2}$$

$$= \tfrac{1}{2} \int_{N-1/2}^{\infty} \frac{dt}{(t+m)(t+n)}$$

by retracing our early steps.

If we let $L = [N - \frac{1}{2}]$, we have

$$\int_{N}^{\infty} \left(\sum_{n=0}^{M} a_n \frac{\sin \pi t}{t+n} \right)^2 dt < \tfrac{1}{2} \sum_{n=0}^{M} \sum_{n=0}^{M} a_m a_n \int_{L}^{\infty} \frac{dt}{(t+m)(t+n)}$$

$$= \tfrac{1}{2} \sum_{m=L}^{M+L} \sum_{n=L}^{M+L} a_{m-L} a_{n-L} \int_{0}^{\infty} \frac{dt}{(t+m)(t+n)}$$

But

$$k(x, y) = \int_{0}^{\infty} \frac{dt}{(t+x)(t+y)} = \frac{\ln x/y}{x - y}$$

satisfies the hypotheses of the preceding version of the Hilbert Double Series Theorem. We can therefore proceed directly, and obtain

$$\int_{N}^{\infty} \left(\sum_{n=0}^{M} a_n \frac{\sin \pi t}{t+n} \right)^2 dt < \sum_{0}^{M} a_n^2 \int_{1}^{\sqrt{(M+[N-1/2]+1/2)/([N-1/2]-1/2)}} \frac{\ln y \, dy}{(y-1)\sqrt{y}}; \tag{4.2}$$

or we can first use the inequality

$$\sum \sum a_m a_n \frac{\log (m/n)}{m - n} \leqslant \pi \sum \sum \frac{a_m a_n}{m + n} \tag{4.3}$$

which is given in Hardy *et al*, [1], p. 354. The latter yields less sharp, but simpler formulas. We obtain, since $1/(x + y)$ also satisfies the hypotheses of our theorem, that

$$\int_N^\infty \left(\sum_{n=0}^M a_n \frac{\sin \pi t}{t+n}\right)^2 dt < \pi \sum_0^M a_n^2 \int_1^{\sqrt{(M+[N-1/2]+1/2)/([N-1/2]-1/2)}} \frac{dy}{(1+y)\sqrt{y}}$$

$$= 2\pi \sum_0^M a_n^2 \left[\arc\tan \sqrt[4]{\frac{M+[N-\frac{1}{2}]+\frac{1}{2}}{[N-\frac{1}{2}]-\frac{1}{2}}} - \frac{\pi}{4}\right]$$

$$= \sum_0^M a_n^2 \left[\frac{\pi^2}{2} - \arc\tan \sqrt[4]{\frac{[N-\frac{1}{2}]-\frac{1}{2}}{M+[N-\frac{1}{2}]+\frac{1}{2}}}\right]. \tag{4.4}$$

Now (4.2) yields the upper bound, for any $f \in S_M$,

$$\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} < \frac{1}{2\pi^2}\frac{M+1}{N-1}, \tag{4.5}$$

which is good if M/N is going to 0. On the other hand, (4.4) yields the upper bound, for $f \in S_M$,

$$\frac{\int_N^\infty f^2(t)\,dt}{\int_{-\infty}^\infty f^2(t)\,dt} < \tfrac{1}{2} - \frac{1}{4\pi}\sqrt[4]{\frac{N-2}{M+N}}, \tag{4.6}$$

which is good if M/N is becoming large.

To complete our knowledge we still need lower bounds to go with (4.2) and (4.6). We begin similarly to Theorem 1(b), by guessing $a_n = 1/\sqrt{L+n}$ where L is the next integer $\geqslant N + \frac{1}{2}$; since we are dealing with *finite* sampling series this time, there is no problem of square-integrability, but our bounds will be more difficult to derive.

First of all,

$$\sum_0^M a_n^2 = \sum_L^{M+L} \frac{1}{n} < \int_L^{M+L} \frac{dx}{x} + \frac{1}{L} = \log\frac{M+L}{L} + \frac{1}{L}.$$

On the other hand,

$$\int_N^\infty \left(\sum_{n=0}^M a_n \frac{\sin \pi t}{t+n}\right)^2 dt = \sum_{n=0}^M \sum_{m=0}^M a_n a_m \int_N^\infty \frac{\sin^2 \pi t}{(t+m)(t+n)} dt$$
$$> \sum_{n=0}^M \sum_{m=0}^M a_n a_m \tfrac{1}{2} \int_{N+1/2}^\infty \frac{dt}{(t+m)(t+n)}$$
$$> \tfrac{1}{2} \sum_{n=L}^{M+L} \sum_{m=L}^{M+L} a_{n-L} a_{m-L} \int_0^\infty \frac{dt}{(t+m)(t+n)}$$

where L is the next integer $\geqslant N + \frac{1}{2}$. In our specific case, the latter equals

$$\tfrac{1}{2} \sum_{n=L}^{M+L} \sum_{m=L}^{N+L} \frac{1}{\sqrt{n}} \frac{1}{\sqrt{m}} \int_0^\infty \frac{dt}{(t+m)(t+n)} > \tfrac{1}{2} \int_L^{M+L} \int_L^{M+L} \frac{1}{\sqrt{xy}} \frac{\ln y/x}{y-x} dy\, dx \quad (4.7)$$
$$= \tfrac{1}{2} \int_L^{M+L} \frac{dx}{x} \int_{L/x}^{(M+L)/x} \frac{\ln u\, du}{(u-1)\sqrt{u}}$$
$$\text{if} \qquad u = y/x.$$

We can give a simple lower bound right away. For as x varies between L and $M + L$, the *minimum* of

$$H(x) = \int_{L/x}^{(M+L)/x} \frac{\ln u\, du}{(u-1)\sqrt{u}}$$

comes at the end points. We saw, in fact, in the proof of our version of the Hilbert Double Series Theorem, that $H(x)$ has a maximum at $x = \sqrt{L(M+L)}$ and no other stationary points in the interval. Hence we have a lower bound of

$$\tfrac{1}{2} \int_L^{M+L} \frac{dx}{x} \int_1^{(M+L)/L} \frac{\ln u\, du}{\sqrt{u}(u-1)} = \tfrac{1}{2}\left(\log \frac{M+L}{L}\right) \int_1^{(M+L)/L} \frac{\ln u\, du}{\sqrt{u}(u-1)} \quad (4.8)$$

Thus, for this choice of a_n,

$$\frac{\int_N^\infty f^2(t)\, dt}{\int_{-\infty}^\infty f^2(t)\, dt} > \frac{1}{2\pi^2} \int_1^{(M+L)/L} \frac{\ln u\, du}{\sqrt{u}(u-1)} \cdot \frac{\ln (M+L)/L}{\ln (M+L)/L + 1/L}$$

where L is the next integer after $N + \frac{1}{2}$. This bound is fine if $M/N \to 0$, or remains bounded; as $M/N \to \infty$, however, it approaches *half* the upper bound (4.6), which does not tell us if $\frac{1}{2}$ is the correct limiting value. We thus have to handle (4.7) much more carefully if $M/N \to \infty$. We begin by writing

$$\int_{L/x}^{(M+L)/x} \frac{\ln u\, du}{\sqrt{u}(u-1)} = \int_0^{\infty} - \int_0^{L/x} - \int_{(M+L)/x}^{\infty} \frac{\ln u\, du}{\sqrt{u}(u-1)}.$$

$$\int_0^{\infty} \frac{\ln u\, du}{\sqrt{u}(u-1)} = \pi^2, \tag{1}$$

so that this contributes

$$\tfrac{1}{2}\pi^2 \log \frac{M+L}{L}$$

to (4.7).

$$\int_0^{L/x} \frac{\ln u\, du}{\sqrt{u}(u-1)} < \max_{0 \leqslant u < \infty} \left(\frac{u^{1/4} \ln u}{u-1}\right) \int_0^{L/x} \frac{du}{u^{3/4}} = 4k_1 \frac{L^{1/4}}{x^{1/4}}, \tag{2}$$

therefore,

$$\tfrac{1}{2} \int_L^{M+L} \frac{dx}{x} \int_0^{L/x} \frac{\ln u\, du}{(u-1)\sqrt{u}} < 2k_1 L^{1/4} \int_L^{M+L} \frac{dx}{x^{5/4}} = 8k_1\left(1 - \left(\frac{L}{M+L}\right)^{1/4}\right).$$

We have naturally saved the hardest until last.

$$\int_{(M+L)/x}^{\infty} \frac{\ln u\, du}{(u-1)\sqrt{u}} < \int_{(M+L)/x}^{\infty} \frac{\ln u\, du}{(u-1)^{3/2}} \tag{3}$$

$$= 2\frac{\ln (M+L)/x}{[(M+L/x)-1]^{1/2}} + 2\int_{(M+L)/x}^{\infty} \frac{du}{u(u-1)^{1/2}}$$

by partial integration. The latter equals

$$2\frac{\ln (M+L)/x}{[(M+L)/x-1]^{1/2}} + 4 \arctan \sqrt{\frac{x}{M+L-x}}$$

$$< 2\frac{\ln (M+L)/x}{[(M+L)/x-1]^{1/2}} + 4\sqrt{\frac{x}{M+L-x}}.$$

Therefore

$$\frac{1}{2}\int_{L}^{M+L}\frac{dx}{x}\int_{(M+L)/x}^{\infty}\frac{\ln u\,du}{(u-1)\sqrt{u}}$$

$$< \int_{L}^{M+L}\frac{1}{x}\frac{\ln (M+L)/x}{\sqrt{(M+L)/x-1}}\,dx + 2\int_{L}^{M+L}\frac{dx}{\sqrt{x(M+L-x)}}$$

$$= \int_{1}^{(M+L)/x}\frac{dy\ln y}{y\sqrt{y-1}} + 2\left[\frac{\pi}{2} - \arcsin\sqrt{\frac{L}{M+L}}\right]$$

$$< \int_{1}^{\infty}\frac{dy\ln y}{y\sqrt{y-1}} + 2\left[\frac{\pi}{2} - \arcsin\sqrt{\frac{L}{M+L}}\right]$$

$$= 2\pi\ln 2 + \pi - 2\arcsin\sqrt{\frac{L}{M+L}}$$

$$< k_2 - 2\sqrt{\frac{L}{M+L}},$$

since $\arcsin x > x$. Combining our results, we have

$$\frac{1}{2}\int_{L}^{M+L}\frac{dx}{x}\int_{L/x}^{(M+L)/x}\frac{\ln u\,du}{\sqrt{u}\,(u-1)} \geqslant \frac{\pi^2}{2}\log\frac{M+L}{L} - k_3,$$

so that, for $f\in S_M$

$$\frac{\int_{N}^{\infty} f^2(t)\,dt}{\int_{-\infty}^{\infty} f^2(t)\,dt} < \frac{\pi^2/2\log (M+L)/L - k_3}{\log (M+L)/L + 1/L} \tag{4.9}$$

$$= \frac{\pi^2}{2} - \frac{k_3 + \pi^2/2L}{\log (M+L)/L + 1/L}$$

$$\geqslant \frac{\pi^2}{2} - \frac{k}{\log (M+N)/(N+2)} \qquad \text{since} \qquad 0 < \frac{1}{L} \leqslant 1.$$

If $M/N \to \infty$, this last expression approaches, once again, $\pi^2/2$.

Relations (4.2), (4.6), (4.8), and (4.9) combine to yield Theorems 5 and 6 as quoted in the summary.

We come now to considering further integrability restrictions on $f(t)$. The basic theorem is the following.

THEOREM 7: *If, in addition to* $\Sigma_0^\infty |f(-n)|^2 < \infty$, *we know that* $\Sigma_0^\infty |f(-n)|^k < \infty$ *for some* $k < 2$, *then*

$$\int_{N+1/2}^{\infty} f^2(t)\,dt \leqslant C(k)\,\frac{1}{[N]^{(2-k)/2k}}\left(\sum_0^\infty |f(-n)|^2\right)^{1/2}\left(\sum_0^\infty |f(-n)|^k\right)^{1/k},$$

where $C(k)$ *depends only on* k.

PROOF:

$$\int_{N+1/2}^{\infty} f^2(t)\,dt \leqslant \sum_{m=0}^{\infty}\sum_{m=0}^{\infty} a_m a_n \int_{N+1/2}^{\infty} \frac{\sin^2 \pi t}{(t+m)(t+n)}\,dt \quad \text{it} \quad a_n = \frac{1}{\pi}|f(-n)|$$

$$< \tfrac{1}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m a_n \int_{[N]}^{\infty} \frac{dt}{(t+m)(t+n)}.$$

For convenience, let us call $[N] = L$. Then

$$\int_N^\infty f^2(t)\,dt < \tfrac{1}{2}\sum_{m=L}^{\infty}\sum_{n=L}^{\infty} a_{m-L}\,a_{n-L}\int_0^\infty \frac{dt}{(t+m)(t+n)}$$

$$< \frac{\pi}{2}\sum_{m=L}^{\infty}\sum_{n=L}^{\infty} a_{m-L}\,a_{n-L}\,\frac{1}{m+n}$$

by Hardy *et al.* [1], p. 354. But then

$$\sum_L^\infty \sum_L^\infty \frac{a_{m-L}\,a_{n-L}}{m+n}$$

$$= \sum_L^\infty \sum_L^\infty \left(\frac{1}{(m+n)^\rho}\,a_m{}^2\right)^{(k-1)/k}\left(\frac{1}{(m+n)^\rho}\,a_n{}^k\right)^{1/2}(a_m{}^2\,a_n{}^k)^{1/2-(k-1)/k},$$

where

$$\rho = \left(\tfrac{1}{2} + \frac{k-1}{k}\right)^{-1}.$$

If we apply the Holder inequality, we obtain

$$\sum_{L}^{\infty} \sum_{L}^{\infty} \frac{a_{m-L}\, a_{n-L}}{m+n}$$

$$\leqslant \left(\sum_{L}^{\infty} \sum_{L}^{\infty} \frac{a_m{}^2}{(m+n)^\rho} \right)^{(k-1)/k} \left(\sum_{L}^{\infty} \sum_{L}^{\infty} \frac{a_n{}^k}{(m+n)^\rho} \right)^{1/2} \left(\sum_{L}^{\infty} \sum_{L}^{\infty} a_m{}^2 a_n{}^k \right)^{1/2-(k-1)/k}$$

$$\leqslant \left(\sum_{L}^{\infty} \frac{1}{(1+m)^\rho} \right)^{(k-1)/k+1/2} \left(\sum_{L}^{\infty} a_m{}^2 \right)^{1/2} \left(\sum_{L}^{\infty} a_n{}^k \right)^{1/k}$$

since $1/(m+n)^\rho < 1/(1+m)^\rho$. Then

$$\sum_{L}^{\infty} \frac{1}{(1+m)^\rho} < \int_{L}^{\infty} \frac{1}{x^\rho}\, dx = \frac{1}{(\rho - 1) L^{\rho-1}},$$

so that

$$\sum_{L}^{\infty} \sum_{L}^{\infty} \frac{a_{m-L}\, a_{n-L}}{m+n} \leqslant \frac{1}{(\rho-1)^{1/\rho}} \frac{1}{L^{1-1/\rho}} \left(\sum a_m{}^2 \right)^{1/2} \left(\sum a_n{}^k \right)^{1/k}.$$

Since $1 - 1/\rho = (2-k)/2k$, the theorem is proved, with

$$C(k) = \frac{1}{2\pi} \left(\frac{3u-2}{2-u} \right)^{1/2+(k-1)/k}.$$

Since

$$\sum |f(-n)|^k \leqslant \frac{4}{\pi} e^{k\pi/2} \int_{-\infty}^{\infty} |f(t)|^k\, dt$$

(see Boas [3], Theorem 6.7.15). The previous theorem also holds with integrals instead of sums. We have the following *Corollary*.

$$\int_{N+1/2}^{\infty} f^2(t)\, dt \leqslant C_1(k) \frac{1}{[N]^{(2-k)/2k}} \left[\int_{-\infty}^{\infty} f^2(t)\, dt \right]^{1/2} \left[\int_{-\infty}^{\infty} |f(t)|^k\, dt \right]^{1/k}$$

Finally, let us seek a simple condition on the spectrum $F(x)$ which is sufficient to insure that the energy to the right of N go to O. A uniform

Lipschitz condition on the spectrum could suffice, but is not very graphic. Boundedness of the spectrum does not seem to be enough; we shall assume that $F(x)$ is of *bounded variation*, i.e.,

$$\int_{-\pi}^{\pi} |dF(x)| < V.$$

Then

$$|a_n| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}F(x)\,dx\right| = \left|\frac{1}{2n\pi i}\int_{-\pi}^{\pi} e^{inx}\,dF(x)\right| \leqslant \frac{V}{2n\pi}, \qquad n \neq 0.$$

Hence, if we let $E = \int_{-\infty}^{\infty} f^2(t)\,dt$,

$$\sum_{n=0}^{\infty} |a_n|^k = |a_0|^k + \sum_{n=1}^{\infty} |a_n|^k$$

$$\leqslant \left(\frac{\sqrt{E}}{\pi}\right)^k + \frac{V^k}{(2\pi)^k}\sum_{n=1}^{\infty}\frac{1}{n^k}$$

$$< \frac{1}{\pi^k}E^{k/2} + \frac{V^k}{2k-2}.$$

Then, by Theorem 7,

$$\int_{N+1/2}^{\infty} f^2(t)\,dt \leqslant \frac{E}{2\pi}\left(\frac{3k-2}{2-k}\right)^{1/2+(k-1)k} \frac{1}{[N]^{(2-k)/2k}}\left\{1 + \left(\frac{V}{\sqrt{E}}\right)^k \frac{1}{2k-2}\right\}^{1/k}$$

We are free to pick k anywhere between 1 and 2. We pick

$$k = \frac{\log [N]}{\log [N] - 1}.$$

If, for the sake of a definite constant, we assume $\log [N] \geqslant 4$, we get

$$\int_{N+1/2}^{\infty} f^2(t)\,dt \leqslant C\frac{\log [N]}{\sqrt{[N]}}.$$

where C may be taken as $(3E/2\pi)\,e[1 + (V/\sqrt{E})^{4/3}]$. We have proved

THEOREM 8. *If $f(t) \in S$ with $\int_{-\infty}^{\infty} f^2(t)\,dt \leqslant E$, and in addition the total variation $\int_{-\pi}^{\pi} |dF(x)|$ of the spectrum is bounded by V, then, for large N,*

$$\int_{N}^{\infty} f^2(t)\,dt \leqslant C\,\frac{\log N}{N^{1/2}}\,,$$

where C depends only on E and V.

REFERENCES

1. HARDY, G. H., LITTLEWOOD, J. E., AND POLYA, G. "Inequalities," 2nd ed Cambridge Univ. Press, London and New York, 1954.
2. RIESZ, F., AND SZ.-NAGY, B "Functional Analysis." Frederick Ungar, New York, 1955.
3. BOAS, R. P., JR. "Entire Functions," Pure and Appl Math. Series. Academic Press, New York, 1954